A COMBINATORIAL IDENTITY AND ITS RELATED CONJECTURE

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ABSTRACT. In recent researches on a discriminant for polynomials, I faced with a recursive (combinatorial) sequence $\lambda_{n,m}$ that its first four terms and identities are $\lambda_{0,m} := (m) = (m-1), \lambda_{1,m} := (\begin{pmatrix} m \\ 1 \end{pmatrix}) = (m-1), \lambda_{2,m} := \left(\begin{pmatrix} m \\ 2 \end{pmatrix}\right) = (m+1), \lambda_{3,m} := \left(\begin{pmatrix} m \\ 3 \end{pmatrix}\right) - 2\left(\begin{pmatrix} m \\ 2 \end{pmatrix}\right) + \left(\begin{pmatrix} m \\ 1 \end{pmatrix}\right)$. In this paper I introduce it, prove an identity thereabout, and leave a problem and a conjecture concerning it.

1. INTRODUCTION

In [2], I faced to a new recursive doubled sequence as follows. For every integer numbers $n, m \geq 0$, define the sequence $\lambda_{n,m}$ by the following recursive definition

$$\lambda_{0,m} := 1, \quad \lambda_{n,m} := \sum_{j=0}^{n-1} (-1)^{n+j+1} \binom{m}{n-j} \lambda_{j,m}.$$

Therefore

$$\lambda_{0,m} := \left(\begin{pmatrix} m \\ 0 \end{pmatrix}\right), \lambda_{1,m} := \left(\begin{pmatrix} m \\ 1 \end{pmatrix}\right), \lambda_{2,m} := \left(\begin{pmatrix} m \\ 2 \end{pmatrix}\right) - \left(\begin{pmatrix} m \\ 2 \end{pmatrix}\right),$$

$$\lambda_{3,m} := \left(\begin{pmatrix} m \\ 3 \end{pmatrix}\right) - 2\left(\begin{pmatrix} m \\ 2 \end{pmatrix}\right) + \left(\begin{pmatrix} m \\ 1 \end{pmatrix}\right),$$

$$\lambda_{4,m} := \left(\begin{pmatrix} m \\ 4 \end{pmatrix}\right) - 3\left(\begin{pmatrix} m \\ 2 \end{pmatrix}\right)^2 + 2\left(\begin{pmatrix} m \\ 1 \end{pmatrix}\right)\left(\begin{pmatrix} m \\ 3 \end{pmatrix}\right) + \left(\begin{pmatrix} m \\ 2 \end{pmatrix}\right)^2 - \left(\begin{pmatrix} m \\ 4 \end{pmatrix}\right), \cdots.$$

As regards finding the coefficients of the above summations a problem and related conjecture have been raised. A simplified formula and an explicit definition for $\lambda_{n,m}$ are aimed for this interesting sequence.

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2. Main theorem and problem

Now, we prove the first identity for $\lambda_{n,m}$.

**Theorem 2.1.** For every integer numbers $n, m \geq 0$, we have $\lambda_{n,m} = \binom{m-1+n}{m-1}$.

**Proof.** For $n = 0$ the formula is true. Next, assume that the formula holds for every $1 \leq k \leq n$. Consider the following well-known combinatorial identity (see [1])

$$
\sum_{j=0}^{k} (-1)^j \binom{m}{k-j} \binom{m-1+j}{m-1} = 0
$$

Upon replacing $k$ by $k + 1$ in the above identity we get

$$
\sum_{j=0}^{k} (-1)^j \binom{m}{k+1-j} \binom{m-1+j}{m-1} = (-1)^k \binom{m+k}{m-1}.
$$

Employing the above identity we obtain

$$
\lambda_{n+1,m} = (-1)^n \binom{m}{n+1} + \sum_{j=0}^{n-1} (-1)^{n+j+1} \binom{m}{n-j} \binom{m+j}{m-1}
$$

$$
= (-1)^n \binom{m}{n+1} + (-1)^n (-1)^n \binom{m+n}{m-1} - \binom{m}{n+1} = \binom{m-1+(n+1)}{m-1}.
$$

Therefore, the proof is complete. $\square$

Now, let $m$ be a fixed positive integer. By using the above theorem, we observe that

$$
\binom{m}{0} = \binom{m-1}{m-1},
$$

$$
\binom{m}{1} = \binom{m}{m-1},
$$

$$
\binom{m}{2} - \binom{m}{2} = \binom{m+1}{m-1},
$$

$$
\binom{m}{3} - 2\binom{m}{2} + \binom{m}{3} = \binom{m+2}{m-1},
$$

$$
\binom{m}{4} - 3\binom{m}{3} + 2\binom{m}{2} + \binom{m}{4} = \binom{m+3}{m-1},
$$

$$
\binom{m}{5} - 4\binom{m}{4} + 3\binom{m}{3} - 2\binom{m}{2} + \binom{m}{5} = \binom{m+4}{m-1},
$$

\ldots

In general, we have

$$
\binom{m-1+n}{m-1} = \sum_{n_1+\cdots+n_k=n, \ 0 \leq n_1 \leq \cdots \leq n_k} C_{n_1,\ldots,n_k} \binom{m}{n_1} \cdots \binom{m}{n_k}
$$
\[ i_1 + 2i_2 + \cdots + n i_n = n \]
\[ i_j \geq 0 \]

where \( C_{i_1,\ldots,i_n} \) and \( d_{i_1,\ldots,i_n} \) are integers subject to the next conjecture. Therefore we have the following problem.

**Problem.** Find the values of \( C_{i_1,\ldots,i_n}, d_{i_1,\ldots,i_n} \).

Following the above problem, we guess the conjecture bellow.

**Conjecture.** The values of \( C_{i_1,\ldots,i_n}, d_{i_1,\ldots,i_n} \) are dependent only on \( n \) (independent from \( m \)), \( d_{n,0,\ldots,0} = 1 \), \( d_{0,\ldots,0,1} = (-1)^{n+1} \), and the summation of all coefficients \( d_{i_1,\ldots,i_n} \) is zero for all \( n > 1 \).

**REFERENCES**


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