A surprising regularity in the number of Hamilton paths in polygonal bigraphs

Gustavus J. Simmons

Abstract:
The smallest bigraph which is edge critical but not edge minimal with respect to Hamilton laceability is the Franklin graph. Polygonal bigraphs*, $P_m$, which generalize one of the many symmetries of the Franklin graph, share this property of being edge critical but not edge minimal [1]. An enumeration of Hamilton paths in $P_m$ for small $m$ reveals surprising regularities: there are $2^m$ Hamilton paths between every pair of adjacent vertices, $3 \times 2^{m-1}$ between every vertex and a unique companion vertex and $3 \times 2^{m-2}$ between all other pairs. Hamilton laceability only requires there be at least one Hamilton path between every pair of vertices in different parts; this says there are exponentially many.

Introduction:
Figure 1 shows the most common graphical representation of the Franklin graph and the same graphical representation for $P_3$.

![Franklin graph ($P_3$) and $P_3$](image)

Figure 1

This representation, which lends itself to proving $P_m$ to be edge critical with respect to Hamilton laceability, does not lend itself to the investigation of Hamilton paths in $P_m$. However, an easily described alternate representation which maps $P_m$ onto an annulus does. The $6m$ edges in $P_m$ have a natural partition into the $2m$ edges not on a quadrilateral, $R_m$, and the $4m$ that are, $Q_m$.

* Extend the sides of a regular polygon on $2m$ vertices, $m \geq 2$, to define the $2m(m - 1)$ finite points of intersection. Circumscribe a centrally symmetric circle large enough all of the points of intersection are in its interior. The $4m$ points of intersection of the extended edges of the polygon with the circle are the vertices of the polygonal bigraph, $P_m$. The edges are the $4m$ arcs of the circle between the vertices and the $2m$ diagonals defined by the extended edges. $P_3$ is the edge skeleton of the 3-cube, $Q_3$. $P_3$ is the Franklin graph.

ARS COMBINATORIA 115(2014), pp. 335-341
The graphical representation in Figure 1 further partitions the 4m edges in \( Q_m \) into 2m on the outer cycle, \( C_m \), and the 2m diagonals, \( D_m \). This is an artificial partition since there are mappings of \( P_m \) that carry any edge in \( Q_m \) and its edge incidences into any other edge in \( Q_m \), but is useful in describing the annular mapping. Instead of a single cycle on 4m vertices, define two cycles on 2m vertices each. Start with an arbitrary edge in \( R_m \), and form a path using only edges from \( R_m \) and \( D_m \). If m is odd, the path will close to form a cycle with 2m edges. Form another cycle on the remaining 2m edges in \( R_m \) and \( D_m \). Construct a new graphical representation of \( P_m \) with one of the cycles symmetrically enclosing the other — it is immaterial which is the outer cycle and which the inner. If the pairs of edges that were in a quadrilateral in the Figure 1 representation of \( P_m \) are rotated to be in the same relative position in the two cycles, the edges in \( C_m \) will cross connect to the edges in \( D_m \) to form m twisted quadrilaterals lying on the annulus defined by the two concentric cycles. The same annular representation is formally possible when m is even — even though the path constructed using only edges from \( R_m \) and \( D_m \) lies on all 4m edges instead of breaking up into two cycles on 2m. Figure 2 shows the annular representation of \( P_m \).

Figure 2

The construction in Figure 2 suggests a general method for constructing and/or counting Hamilton paths in \( P_m \). While the number of quadrilaterals, m, can be arbitrarily large, at most two can host endpoints for a Hamilton path, which means that for all \( m \geq 3 \) there will be runs of quadrilaterals not containing an endpoint. If the endpoints of the Hamilton path are in the same or adjacent quadrilaterals in \( P_m \), there will be only one such run, while if the endpoints are in non-adjacent quadrilaterals there will be two. Simple parity says that either all four edges from \( R_m \) are used to connect a run to the rest of \( P_m \), or else just one on each end of the run. In either case the path(s) in the run must connect from one end of the run to the other. Since all of the quadrilaterals are twisted, for a path to reverse direction it would have to lie on three of the vertices in some quadrilateral, leaving the fourth vertex isolated from being in a Hamilton path.
There are a couple of simple observations. A path through a run of \( n \) quadrilaterals can start on either cycle and end on either the same cycle or the other. If there is only one path it must span all \( 4n \) vertices. If there are two paths they must each lie on \( 2n \) vertices. The number of Hamilton consistent paths through a run grows exponentially with \( n \) but a simple technique allows them to be succinctly described and enumerated.

There are only six ways paths can lie on all four vertices in a quadrilateral, i.e. for them to be Hamilton consistent; see Figure 3. The quadrilateral path(s) represented by symbols A, C and C’ switch from one cycle to the other. The path(s) represented by symbols B, D and D’ do not.

![Figure 3](image)

The edge(s) from \( R_n \) leaving one quadrilateral must match those entering the next, so there are always exactly two compatible choices for the paths in the next quadrilateral. Table 1 gives the syntactical rules for constructing Hamilton consistent symbol sequences.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Successor symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A or B</td>
</tr>
<tr>
<td>B</td>
<td>A or B</td>
</tr>
<tr>
<td>C</td>
<td>C’ or D’</td>
</tr>
<tr>
<td>C’</td>
<td>C or D</td>
</tr>
<tr>
<td>D</td>
<td>C or D</td>
</tr>
<tr>
<td>D’</td>
<td>C’ or D’</td>
</tr>
</tbody>
</table>

Table 1

The important thing to notice in Table 1 is that all symbols have one successor that switches cycles for the path(s) and one that doesn’t. Therefore the \( 2^{n-1} \) syntactically correct sequences, starting from an arbitrary symbol, partition into \( 2^{n-2} \) that switch cycles and \( 2^{n-2} \) that do not depending on whether an odd or even number of the symbols A, C or C’ occur in the sequence. All that matters, so far as Hamilton paths are concerned, is which of the edges from \( R_n \) connect the run to the rest of \( P_m \) and how. There are six ways this can be done. If the connection uses both edges on each end of the run, the paths through the run can either switch cycles or not. If only a
single edge is used at each end, it can enter on either cycle and exit on either that same cycle or the other. The argument just given shows that there are \(2^{m^2}\) path(s) in each of these six cases. The proofs that follow depend critically on this result.

As an illustration of the power of this result, it is little more than a remark now to show that there are \(3 \times 2^{m-1}\) Hamilton cycles in \(P_m\). As noted earlier the path cannot double back on itself so must encircle the annulus. This says an \(m\) symbol sequence representing a Hamilton cycle must also close on itself to form a cycle. A cyclical AB sequence must have an odd number of occurrences of A so there are \(2^{m-1}\) such cycles. A CD sequence can start with any one of the four symbols from which \(2^{m-1}\) syntactically correct \(m\) symbol sequences originate. But only half of these end in a symbol that can precede the starting symbol. Therefore there are \(4 \times 2^{m-1} \times 1/2 = 2^m\) CD cycles and hence a total of \(3 \times 2^{m-1}\) Hamilton cycles in \(P_m\).

Hamilton paths in \(P_m\):

The problem of counting Hamilton paths between specified endpoints in \(P_m\) reduces to characterizing the ways in which initial paths from the endpoints can connect to runs of quadrilaterals – and then using the properties of runs just developed to describe all possible Hamilton paths. There are three basic cases which must be considered, each of which has further subdivisions depending on in which cycles the endpoints are located.

**Case 1.** The endpoints are in the same quadrilateral

i. The endpoints are on the same cycle

ii. The endpoints are on different cycles

**Case 2.** The endpoints are in adjacent quadrilaterals.

i. The endpoints are on the same edge in \(R_m\) connecting the host quadrilaterals.

ii. The endpoints are on distinct edges in \(R_m\) each of which connects the host quadrilaterals.

iii. The endpoints are on distinct edges in \(R_m\), neither of which connects the two host quadrilaterals.

**Case 3.** The endpoints are in non-adjacent quadrilaterals.

**Case 1.**

Figure 4 shows the eight possible quadrilateral paths; the upper four for sub-case 1i and the lower four for sub-case 1ii. The left hand pair of paths must be connected by an AB sequence which must switch cycles in subcase 1i and not in sub-case 1ii. The remaining quadrilateral paths must be connected by CD sequences. Consequently each of the eight quadrilateral paths in Figure 4 contributes \(2^{m^2}\) Hamilton paths for a total of \(2^m\) in each sub-case.
Case 2.
The subcases must be treated separately since the paths connecting the endpoints to runs differ so greatly. Figure 5 shows the eight possible quadrilateral paths for subcase 2i. Four must be connected by AB sequences and four by CD sequences. Two of the AB sequences require the cycles to be switched and two do not, therefore each of the eight quadrilateral paths contributes $2^m$ Hamilton paths for a total of $2^m$.

Since the only way a pair of endpoints can be adjacent in $P_n$ is to either be in the same quadrilateral or else to be endpoints of an edge in $R_m$ this completes the proof that there are $2^m$ Hamilton paths between every pair of adjacent vertices in $P_n$.

Figure 6 shows the eight possible quadrilateral paths for subcase 2ii. Just as in subcase 2i, four are connected by AB sequences and four by CD sequences. However the four connected by AB sequences this time all have a path reversal so all AB sequences will work. Therefore the total number of Hamilton paths between the endpoints in subcase 2ii is $4x2^{m-2} + 4x2^{m-3} = 3x2^{m-1}$. Figure 2 shows that the 2m edges in $R_m$ are paired through connecting the same pair of quadrilaterals. Since all 4m vertices are in $R_m$, the companion vertex to any vertex is defined by the conditions for subcase 2ii.
Figure 7 shows the six possible quadrilateral paths for subcase 2iii when the endpoints are on the same cycle and Figure 8 for when they are not. The proof argument is the same for both subcases. All twelve paths must be connected by CD sequences, each of which contributes $2^{m^3}$ Hamilton paths. Therefore the total number of Hamilton paths for either vertex pair is $3 \times 2^{m^3}$.

![Figure 7](image)

Case 3. The endpoints are in non-adjacent quadrilaterals.

The endpoints can be in the same cycle or in different cycles. With no loss of generality assume one is in the outer cycle. There are only three Hamilton consistent paths through the host quadrilateral for that choice of an endpoint; see Figure 9.

![Figure 9](image)

There will also be three equivalent Hamilton consistent paths through the other host quadrilateral irrespective of the cycle on which the endpoint lies. It is easy to see
from Figure 9 and the fact that the endpoints are by parity necessarily proximate in
the annulus that the run connecting the endpoint sides of the two host quadrilaterals
will be a CD sequence and the other run an AB sequence. This observation is
crucial to the proof argument.

Let X', Y' and Z' denote the mapping of X, Y and Z into the other host
quadrilateral. There are nine pairings of these quadrilateral paths. The five pairings
that include either X or X' must be treated separately due to the path reversal in X
and X'. The pair X-X' cannot have a Hamilton path since the two path reversals
form a closed loop. The other four pairings do. Assume there are i ≥ 1
quadrilaterals in the AB run. The path reversal at X (or X') forms a loop. Each
occurrence of A in the sequence switches cycles, but the endpoints remain
connected by the loop. Therefore for Hamilton path considerations it doesn't matter
how many A cycle switches occur. As remarked earlier, in this case there are 2'
paths through the AB sequence. There are m - 2 - i quadrilaterals in the CD run, and
consequently 2^{m - 2 - i} Hamilton consistent paths associated with each pair of the
quadrilateral paths. The paths in the two runs are independent, so the total number
is multiplicative; 2x 2^{m - 2 - i} which when multiplied by four, the number of pairs,
yields the result that there are 2^{m-1} Hamilton paths in cases in which one of the
quadrilateral paths is either X or X'. The cases in which neither X nor X' appear
require the AB sequence to either switch cycles or else to not switch them which
results in only 2^{m-2} paths through the AB run and 2^{m-2} paths in all. Summing these
two values shows there are 3x2^{m-1} Hamilton paths between any pair of vertices in
different parts in non-adjacent quadrilaterals.

In summary, there are 2^{m} Hamilton paths between any of the 6m pairs of
adjacent vertices, 3x2^{m-1} between any of the 2m companion vertex pairs defined in
subcase 2ii and 3x2^{m-2} between all of the other 4m^2-8m vertex pairs in Pm.

Concluding remark:

Pm is edge critical with respect to Hamilton laceability [1]. It is truly
surprising, given that there are exponentially many Hamilton paths between every
pair of vertices from different parts in Pm, that deleting an arbitrary edge results in
at least one pair having none.

References:
1. G. J. Simmons, A family of edge critical, but not edge minimal, Hamilton
laceable bigraphs, pending publication Bulletin of the Institute of Combinatorics
and its Applications

341